THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4030 Differential Geometry
19 September, 2024 Tutorial

Definition. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$ there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

numbered differently in Lecture Notes

1. $\mathbf{x}$ is differentiable, that is, if we write
$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U,$$
the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in U.

2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous.

3. (The regularity condition.) For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

avoid this situation

Definition. Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined on an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective mapping (that is, it has a nontrivial kernel). The image $F(p) \in \mathbb{R}^m$ of a critical point is called a *critical value* of F. A point of $\mathbb{R}^m$ which is not a critical value is called a *regular value* of F.

In the case of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, to say that df_p is not surjective is equivalent to saying that the partials $f_x = f_y = f_z = 0$ at p. Hence, $a \in f(U)$ is a regular value if and only if f_x, f_y, f_z do not vanish simultaneously at any point in the inverse image.

Proposition 1. *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.*

Proposition 2. *Let $p \in S$ be a point of a regular surface S and let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{x}(U) \subset S$ such that x is differentiable and for each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one. Then if $\mathbf{x}$ is one-to-one, then $\mathbf{x}^{-1}$ is continuous.*

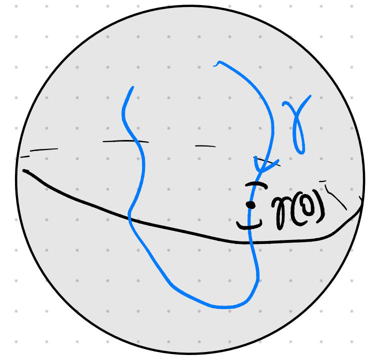
Remarks. It is often easier to show that a set is regular surface by showing it is the inverse image of a regular value by the first proposition above. The above proposition then means that if we already know that S is a regular surface, and we have a candidate x for a parameterization, then we do not need to check that x^{-1} is continuous, as long as the other conditions hold.

Announcement: HW1 due tomorrow (2/19 6pm) on Gradescope

HW Hints:

1.3: How to use Taylor's Theorem:

WLOG it suffices to show γ has positive curvature at $s=0$



Use Taylor's Thm to expand near $\gamma(0)$: $\gamma(s) = \gamma(0) + s\gamma'(0) + s^2\gamma''(0) + O(s^3)$

Use contradiction.

1.4 b/c: $T = \gamma' = b'N + b(-\kappa T + \tau B) + c'B + c(-\tau N)$

Equate coefficients.

1. (From exercise 2-2.7 of [Car16]) Let $f(x,y,z) = (x+y+z-1)^2$.

   (a) Locate the critical points and critical values of f.

   (b) For what values of c are the sets $f(x,y,z) = c$ a regular surface?

a) $f_x = f_y = f_z = 2(x+y+z-1)$, So $\nabla f = 0 \iff z = 1-x-y$

So the set of critical points of f is given by

$\{(x,y,z) : z = 1-x-y\}$.

The set of critical values

$\{(x,y,z) : f(x,y,1-x-y)\}$

$= \{(x,y,z) : (\cancel{x}\cancel{+y}+1\cancel{-x}\cancel{-y}\cancel{-1})^2\} = \{0\}$.

b) As long as $c \neq 0$, $c \in \mathbb{R}$ is a regular value of f and the level sets $f^{-1}(c)$ are regular surfaces.

2. Show that the torus T, generated by rotating a circle of radius r about an axis at fixed distance $a > r$ is a regular surface by showing it is the inverse image of a regular value of a suitably chosen differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$.

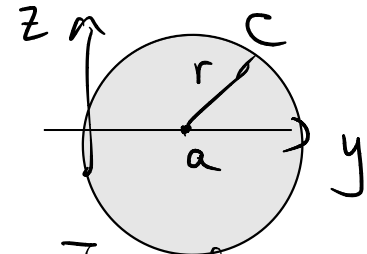
Parameterize Torus as surface of revolution

$$X(u,v) = \big((a+r\cos u)\cos v,\ r\sin v,\ (a+r\cos u)\sin v\big)$$

Need to show X is a parameterization and covers T.

Pf: Let C be a circle of radius r. WLOG, we take C to be in the yz-plane. Then C is given by

$$(y-a)^2 + z^2 = r^2$$



Rotating about the z-axis gives that Torus T satisfies

$$\underbrace{z^2 + \left(\sqrt{x^2+y^2} - a\right)^2}_{f(x,y,z)} = r^2$$

$f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2} - a\right)^2$. Then $T = f^{-1}(r^2)$.

$$f_x = \frac{2x\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}, \quad f_y = \frac{2y\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}, \quad f_z = 2z.$$

So f differentiable except at $(x,y) = (0,0)$.

∇f vanishes only when

1) $\cancel{x = y = z = 0}$

2) $z = 0$, $\sqrt{x^2+y^2} = a$

But $a > r$, so none of these points are in $f^{-1}(r^2)$.

In fact, let $\theta \in [0, 2\pi)$,

$$f(0, a\sin\theta, a\cos\theta) = 0^2 + \left(\sqrt{a^2\sin^2\theta + a^2\cos^2\theta} - a\right)^2 = 0 \neq r^2$$

So r^2 is a regular value of f $\Rightarrow$ T is a regular surface.

3. (Exercise 2-2.4 of [Car16]) Show that for $f(x, y, z) = z^2$, 0 is not a regular value of f, but $f^{-1}(0)$ is a regular surface.

Pf: Clearly $f^{-1}(0)$ is the xy-plane $\Rightarrow$ regular surface.

$g(x,y,z)=z$ would work better

We have $f_x = 0$, $f_y = 0$, $f_z = 2z$.

So ∇f vanishes when $z=0$, so $(x,y,0)$ is a critical pt. of f.

Also $f(x,y,0)=0$, so 0 is a critical value of f.

4. (From exercise 2-2.11 of [Car16]) Show that the set $S = \{(x, y, z) : z = x^2 - y^2\}$ is a regular surface and check that the following is a parameterization for S:

$$\mathbf{x}(u, v) = (u + v, u - v, 4uv), (u, v) \in \mathbb{R}^2$$

Pf: First show S is a regular surface,

$S = f^{-1}(0)$ where $f(x,y,z) = z - x^2 + y^2$ and

0 is a regular value of f.

Check this.

Condition 0: $X(u,v) \in S$: $(u+v)^2 - (u-v)^2 = 4uv$ ✓

Condition 1: $X(u,v) = (u+v, u-v, 4uv)$

differentiable ✓

Condition 3: Compute dx_q for $q \in S$:

$$dx_q = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 4v & 4u \end{bmatrix}$$

and $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0.$

so dx_q is one-to-one.